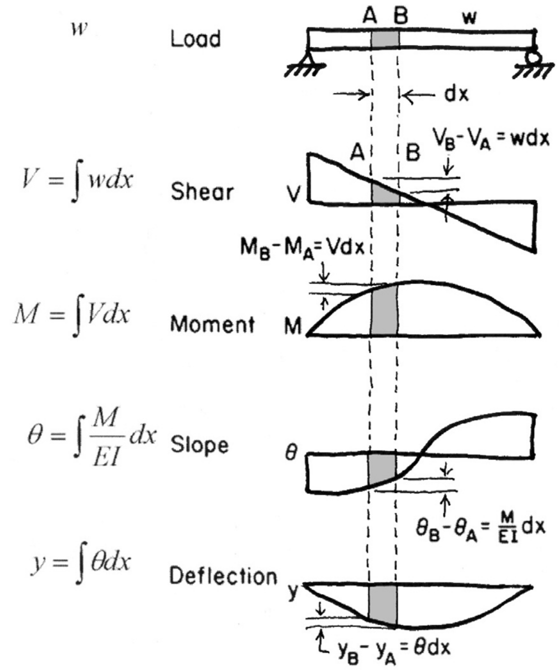
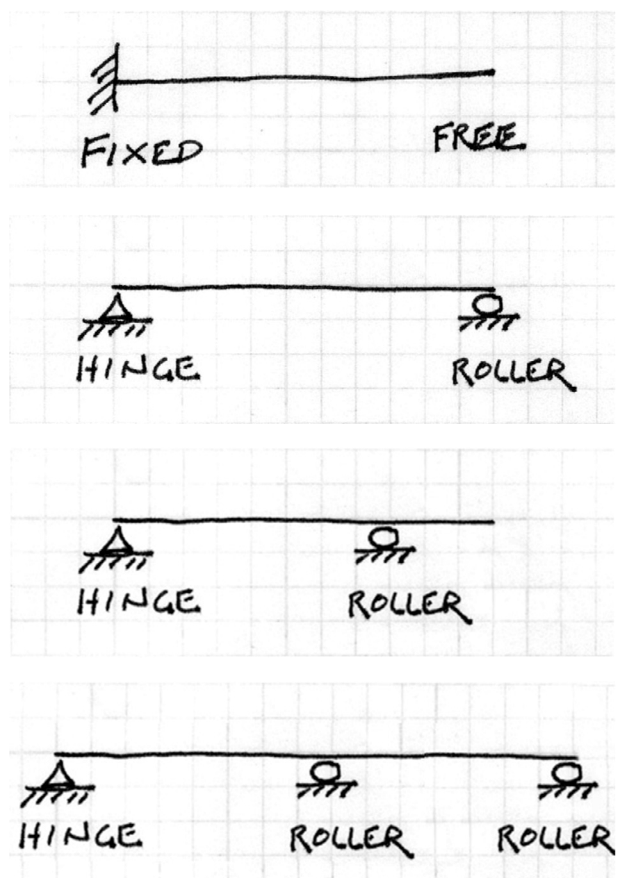
Bending and Shear in Simple Beams

- Free Body Diagrams of Shear and Moment in Beams
- Sign Conventions for Plotting V & M Diagrams
- Diagrams by Equilibrium (FBD)
- Diagrams by Integration
- Diagrams by Areas (Semi-graphical)
- Diagrams by Equations
- Examples in Form (catenary curves)



Beam Types

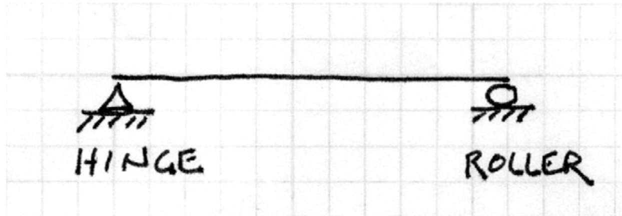
- Cantilever
- Simple
- Simple with Cantilever
- Continuous (multi-span)



Support Conditions

Roller

Fixed in Fy



Hinge (Pinned)

Fixed in Fx

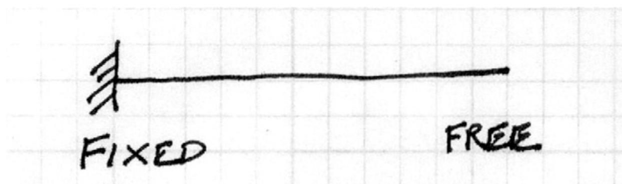
Fixed in Fy

Fixed

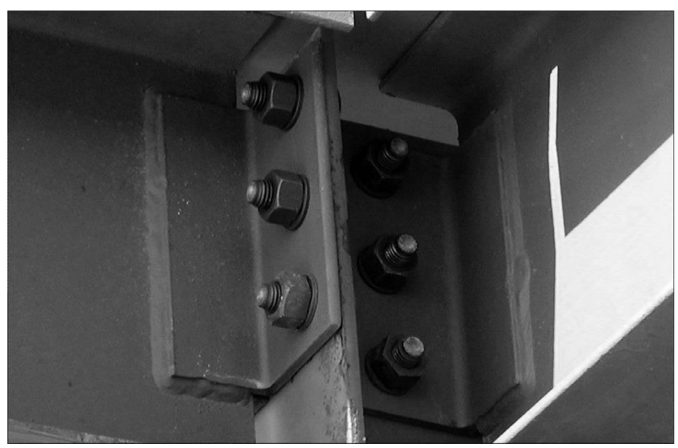
Fixed in Fx

Fixed in Fy

Fixed in Mz



Connection Types



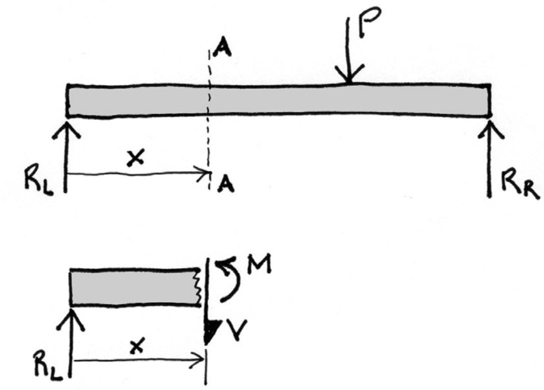
Bearing (or simple)

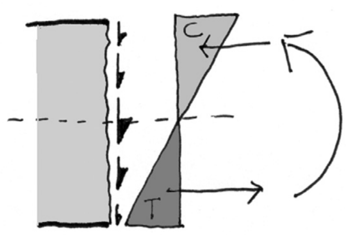
Slip Critical (or fixed)

Internal Shear and Moment

Cutting any section through a beam will reveal internal shear and moment forces necessary to maintain static equilibrium.

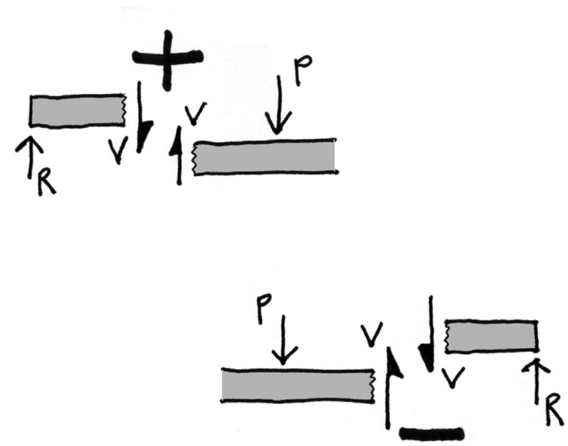
The shears can be determined by summing vertical forces and the moments by summing moments.





Sign Convention for Shear

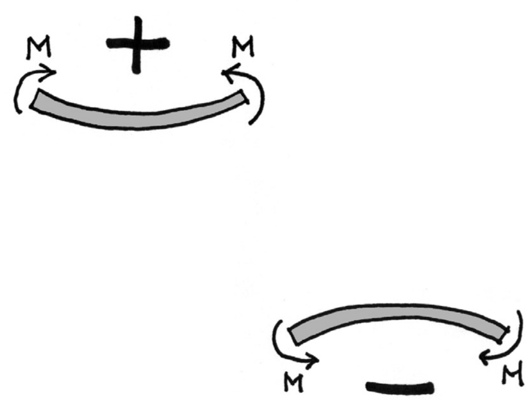
- \+ the sum of the vertical forces to the left of the cut is upwards
- \- the sum of the vertical forces to the left of the cut is downwards



Sign Convention for Moment

- \+ the top fibers are in compression
- \- the top fibers are in tension

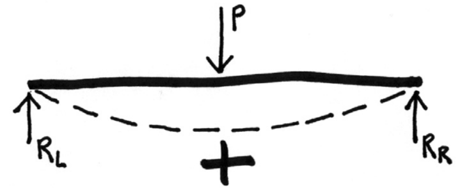
the European moment convention is the reverse

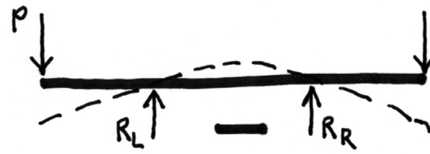


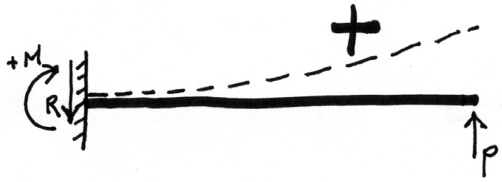
Sign Convention for Moment

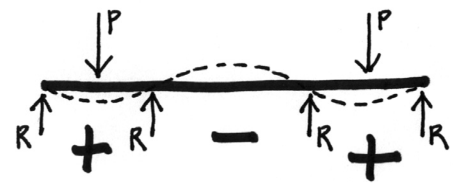
- **+** positive curvature (holds water)
- **-** negative curvature (spills water)

the European moment convention is the reverse









Relationships of Forces and Deformations

There are a series of relationships among forces and deformations in a beam, which can be useful in analysis. Using either the deflection or load as a starting point, the following characteristics can be discovered by taking successive derivatives or integrals of the beam equations.

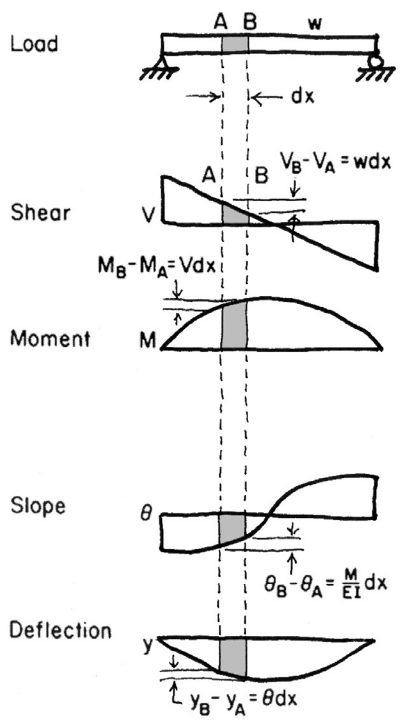
w

$V = \int w dx$

$M = \int V dx$

$\theta = \int \frac{M}{EI} dx$

$y = \int \theta dx$



load $= \frac{d^4y}{dx^4} = \frac{dV}{dx}\frac{1}{EI} = \frac{w}{EI}$

shear $= \frac{d^3y}{dx^3} = \frac{dM}{dx}\frac{1}{EI} = \frac{V}{EI}$

moment $= \frac{d^2y}{dx^2} = \frac{M}{EI}$

slope $= \frac{dy}{dx}$

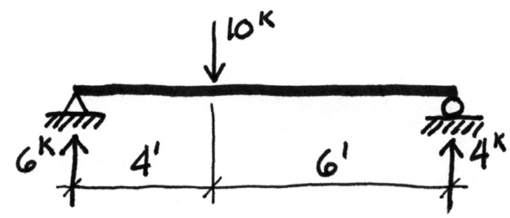
deflection $= y$

Methods to Determine Values of Shear and Moment

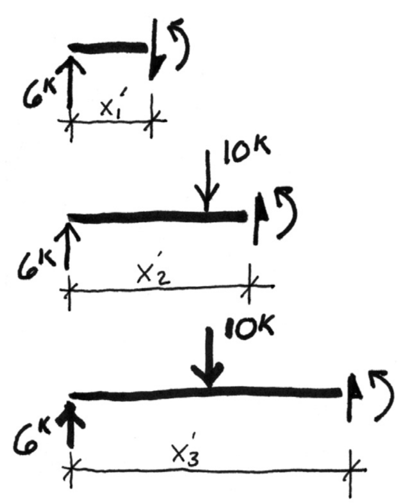
1. Equilibrium Method
    - Select a point along the beam
    - Cut a section and draw the FBD
    - Solve for the internal shear and moment forces at the section
2. Integration of Equations
    - Write the equation of the load function
    - Integrate load equation to get shear equation
    - Solve integration constant (use end reaction)
    - Integrate shear equation to get moment equation
    - Solve integration constant (use point with zero moment, e.g. end point)
3. Semi-graphical Method
    - Draw load diagram and solve end reactions with equilibrium equations.
    - Start at left and construct the shear diagram using point loads and areas on load diagram
    - Calculate areas of shear diagram to find change in value on moment diagram
    - Find points of zero moment to begin moment diagram, e.g. end points
4. Superposition of Equations
    - Break the loading into standard cases
    - Use given equations to solve shear and moment for each case
    - Add the cases to get combined values of original loading

1. Equilibrium Method - procedure

To plot the change of internal shear or moment forces, **a series of sections can be cut** along the beam. The exposed forces can be calculated.

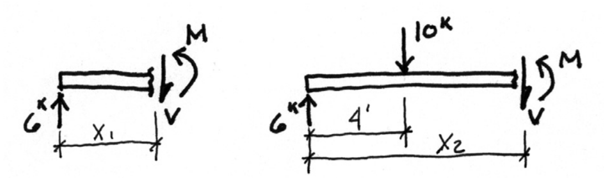


A section should **not be cut "through" an applied force**, but either a bit to the left or to the right of the force.



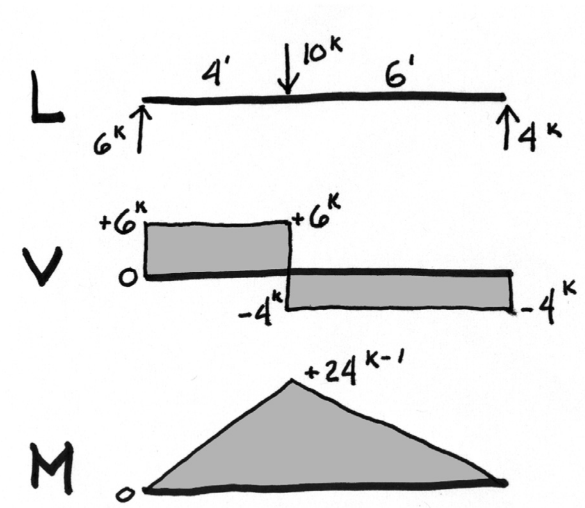
Either the "left" or "right" free body diagram may be used to calculate the forces. The sign convention described earlier must be consistently applied.

1. Equilibrium Method - example



Tabulated Results of FBD Calculations

Cut Location From R_L (ft)	Shear V (k)	Moment M (k-ft)
0-	0	0
0+	6	0
1	6	6
2	6	12
3	6	18
4-	6	24
4+	-4	24
5	-4	20
6	-4	16
7	-4	12
8	-4	8
9	-4	4
10-	-4	0
10+	0	0



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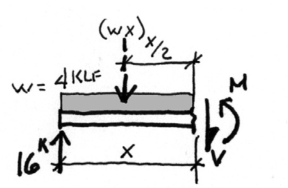

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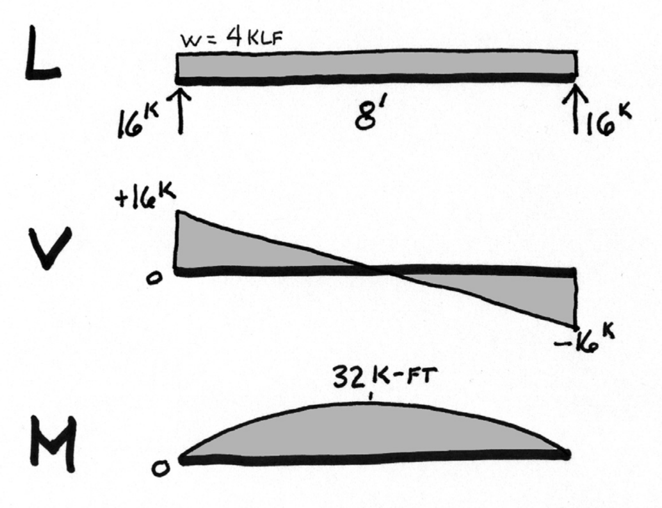
Cut Location From R_L (ft)	Shear V (k)	Moment M (k-ft)
0-	0	0
0+	16	0
1	12	14
2	8	24
3	4	30
4	0	32
5	-4	30
6	-8	24
7	-12	14
8-	-16	0
8+	0	0



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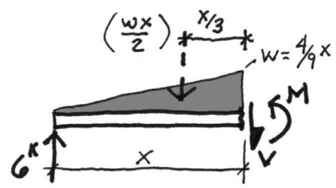
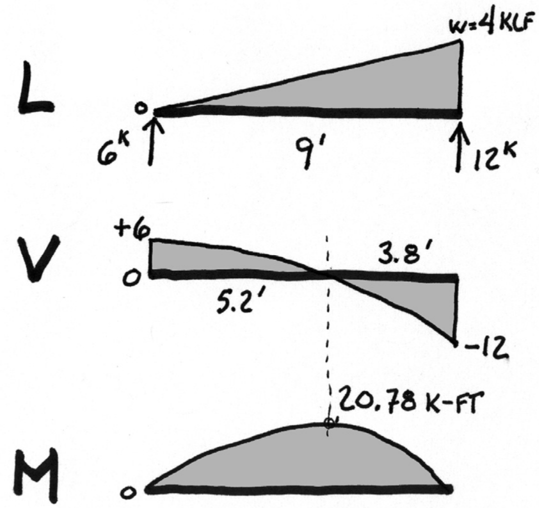


Tabulated Results of FBD Calculations

Cut Location From R_L (ft)	Shear **V** (k)	Moment **M** (k-ft)
0-	0	0
0+	6	0
1	5.78	6.9
2	5.11	11.4
3	4.00	16.0
4	2.44	19.3
5	0.44	20.74
5.2	0	20.78
6	-2.00	20.0
7	-4.90	16.6
8	-8.24	10.0
9-	-12.00	0
9+	0	0



Relationships of Forces and Deformations - procedure

There are a series of relationships among forces and deformations in a beam, which can be useful in analysis. Using either the deflection or load as a starting point, the following characteristics can be discovered by taking successive derivatives or integrals of the beam equations.

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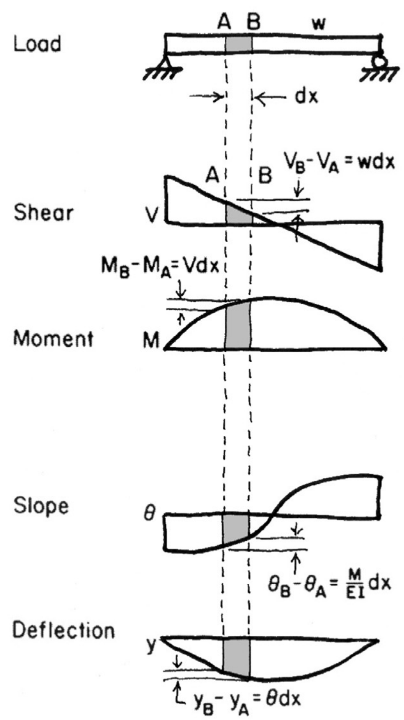
$V = \int w dx$

$M = \int V dx$

$\theta = \int \frac{M}{EI} dx$

$y = \int \theta dx$



load $= \frac{d^4y}{dx^4} = \frac{dV}{dx}\frac{1}{EI} = \frac{w}{EI}$

shear $= \frac{d^3y}{dx^3} = \frac{dM}{dx}\frac{1}{EI} = \frac{V}{EI}$

moment $= \frac{d^2y}{dx^2} = \frac{M}{EI}$

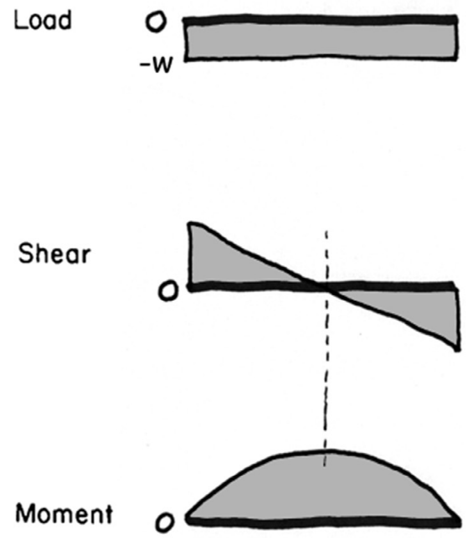
slope $= \frac{dy}{dx}$

deflection $= y$

2. Shear and Moment by Integration - example

One method of solving shear and moment forces is to write the loading equation and solve the integration equations for the shear and moment. One problem using this method can be finding the constant of integration, particularly with discontinuous load functions.

$$-w$$



$$V = \int w dx$$

$$V = -wx + C$$

$$V = -wx + \frac{wl}{2}$$

$$M = \int V dx$$

$$M = -\frac{w}{2}x^2 + \frac{wl}{2}x + C$$

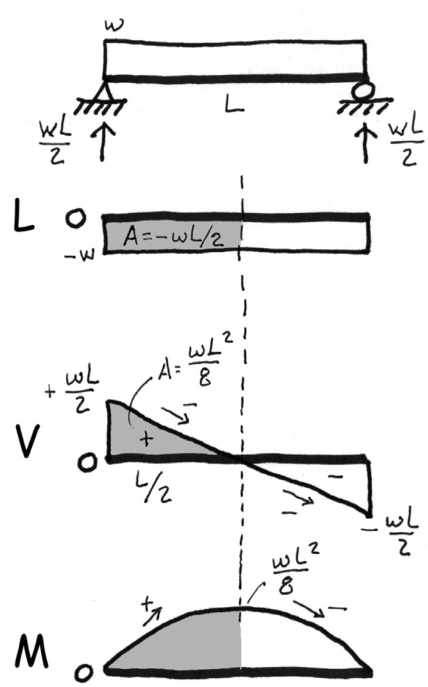
$$M = -\frac{w}{2}x^2 + \frac{wl}{2}x$$

3. Shear and Moment by Semi-graphical Method – diagram relationships

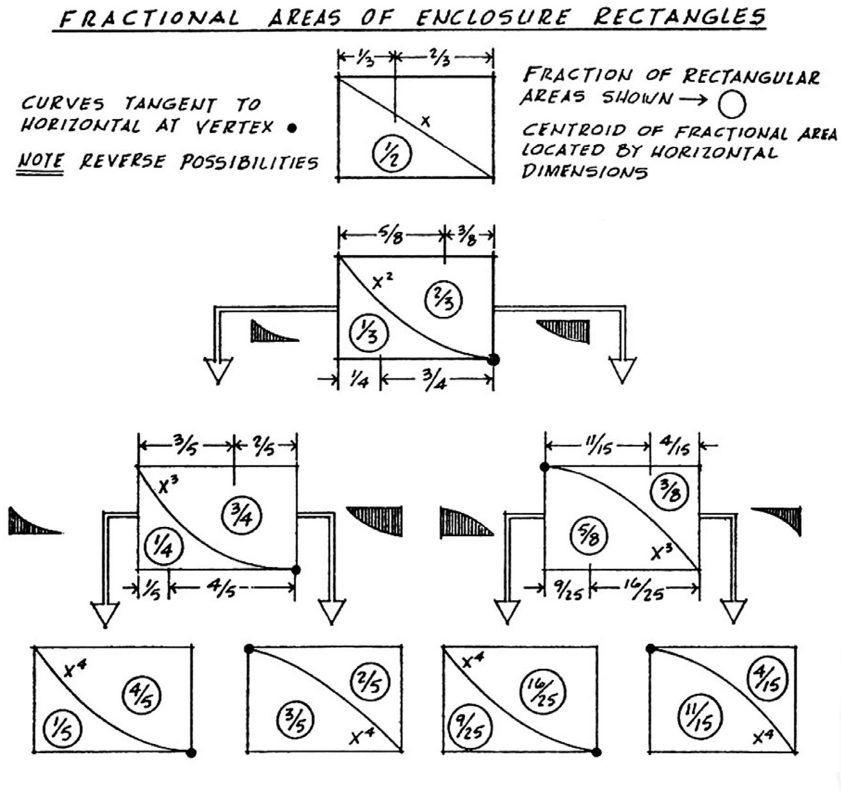
By recognizing the diagrammatic relationships between curves and their derivatives and integrals, shear and moment diagrams can be constructed based on areas and slopes of those curves.

Moving from Upper to Lower Diagrams:

- The area between any two points on the upper diagram is equal to the change in value between same points on the lower diagram.
- The degree of the curve increases by one for each diagram.
- The value on the upper diagram is equal to the slope of the lower diagram.
- Where the upper diagram crosses 0 on the axis, the lower diagram is at a maximum or minimum.
- Points of inflection or "contraflexure" (between + and – curvature) on the elastic curve (deflected shape) are points of zero moment.



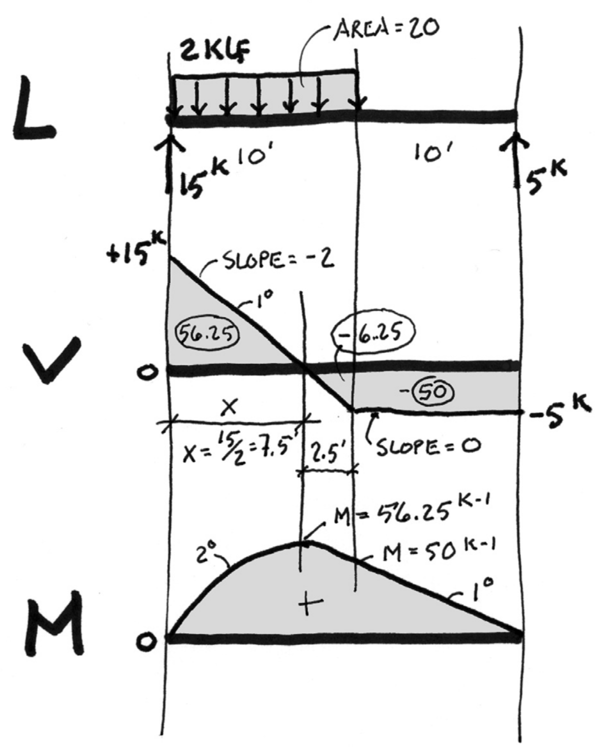
3. Semi-graphical Method



3. Semi-graphical Method

Procedure:

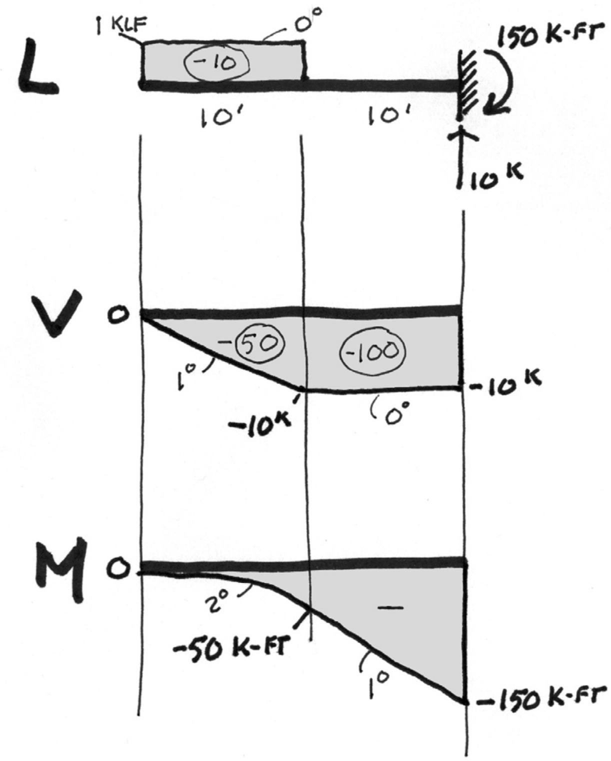
1. Find end reactions
2. Start at left end of V-Diagram and “apply” load from left to right
3. Calculate areas of V-Diagram
4. Find max. and min. values on M-Diagram using V-Diagram areas between axis crossings.
5. Check slope and + or - values



3. Semi-graphical Method

example

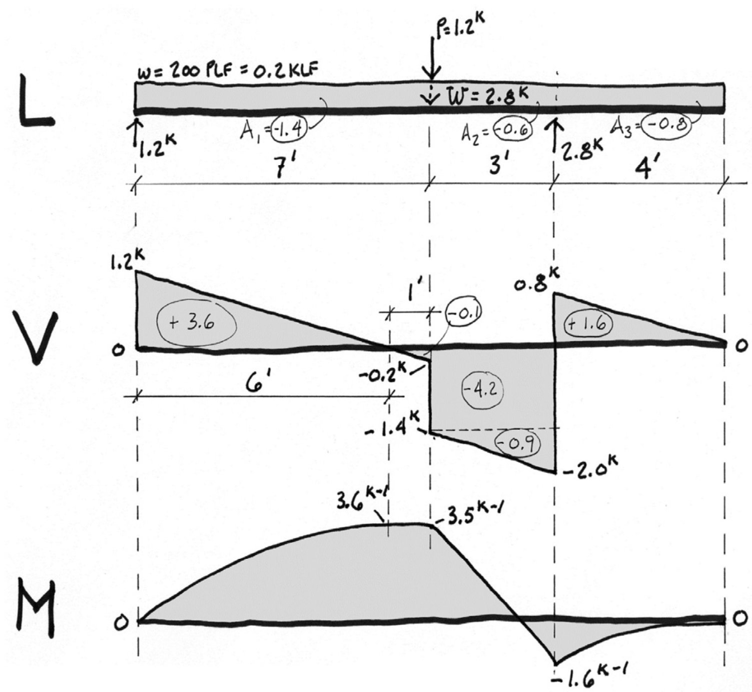
Cantilever Beam



3. Semi-graphical Method

example

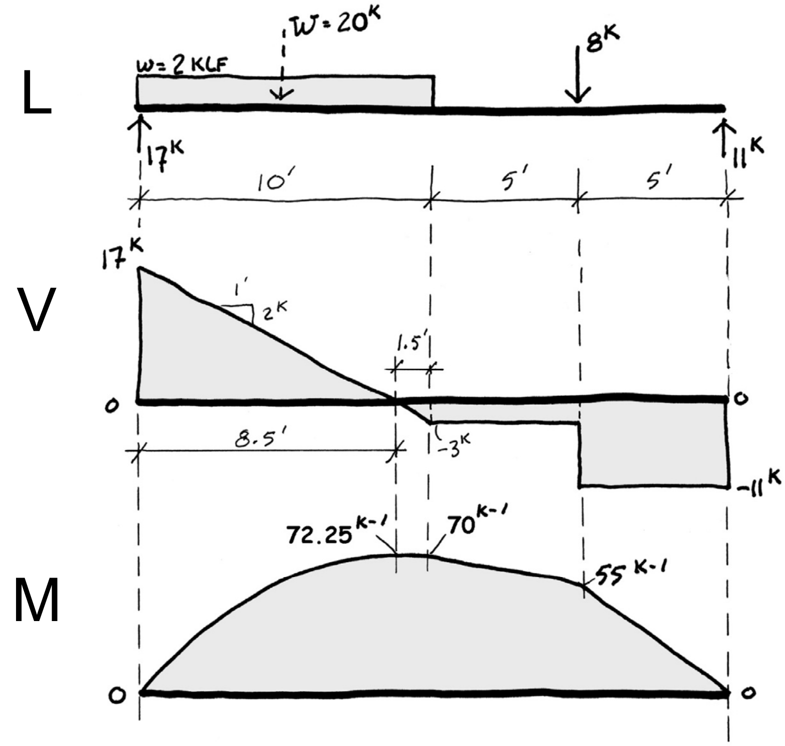
Beam with cantilever



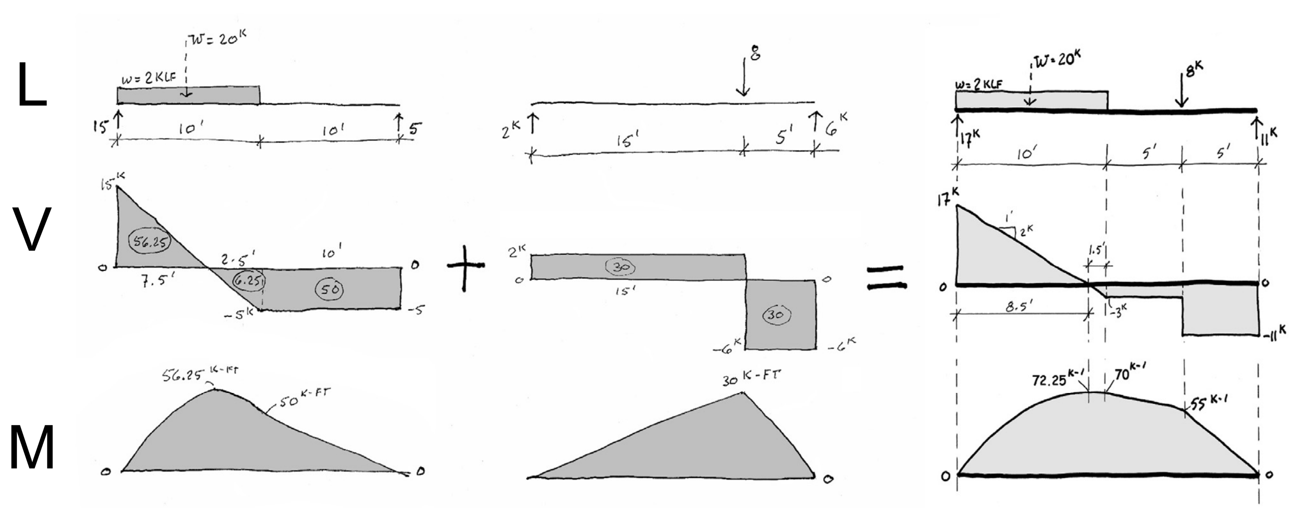
3. Semi-graphical Method

example

Simple beam



3. Semi-graphical Method - Superposition



Equations Method

For simple spans:

V_{max} is the larger reaction

For symmetric loadings:

M_{max} is at C.L.

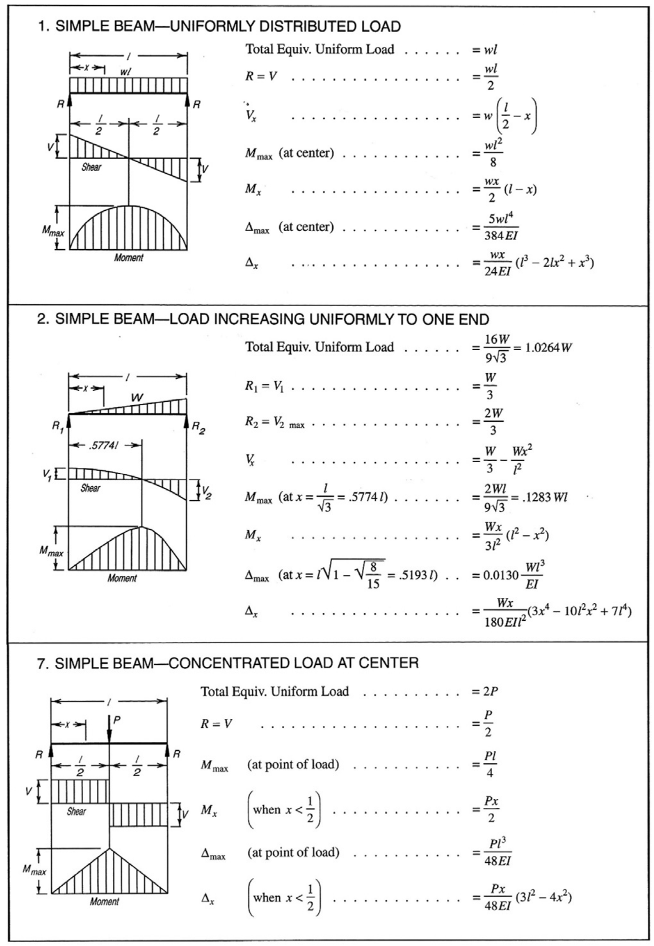
For cantilevers:

Both V_{max} and M_{max} are at the support

In these equations:

w = load per unit length (PLF or KLF)

W = the total load (LB or KIP)



1. SIMPLE BEAM—UNIFORMLY DISTRIBUTED LOAD

Total Equiv. Uniform Load . . . . . . $= wl$

$R = V$ . . . . . . $= \frac{wl}{2}$

V_x . . . . . . $= w\left(\frac{l}{2} - x\right)$

M_{max} (at center) . . . . . . $= \frac{wl^2}{8}$

M_x . . . . . . $= \frac{wx}{2}(l - x)$

Δ_{max} (at center) . . . . . . $= \frac{5wl^4}{384EI}$

Δ_x . . . . . . $= \frac{wx}{24EI}(l^3 - 2lx^2 + x^3)$

2. SIMPLE BEAM—LOAD INCREASING UNIFORMLY TO ONE END

Total Equiv. Uniform Load . . . . . . $= \frac{16W}{9\sqrt{3}} = 1.0264W$

$R_1 = V_1$ . . . . . . $= \frac{W}{3}$

$R_2 = V_{2\,max}$ . . . . . . $= \frac{2W}{3}$

V_x . . . . . . $= \frac{W}{3} - \frac{Wx^2}{l^2}$

M_{max} $\left(\text{at } x = \frac{l}{\sqrt{3}} = .5774l\right)$ . . . . . . $= \frac{2Wl}{9\sqrt{3}} = .1283Wl$

M_x . . . . . . $= \frac{Wx}{3l^2}(l^2 - x^2)$

Δ_{max} $\left(\text{at } x = l\sqrt{1 - \sqrt{\frac{8}{15}}} = .5193l\right)$ . . $= 0.0130\frac{Wl^3}{EI}$

Δ_x . . . . . . $= \frac{Wx}{180EIl^2}(3x^4 - 10l^2x^2 + 7l^4)$

7. SIMPLE BEAM—CONCENTRATED LOAD AT CENTER

Total Equiv. Uniform Load . . . . . . $= 2P$

$R = V$ . . . . . . $= \frac{P}{2}$

M_{max} (at point of load) . . . . . . $= \frac{Pl}{4}$

M_x $\left(\text{when } x < \frac{l}{2}\right)$ . . . . . . $= \frac{Px}{2}$

Δ_{max} (at point of load) . . . . . . $= \frac{Pl^3}{48EI}$

Δ_x $\left(\text{when } x < \frac{l}{2}\right)$ . . . . . . $= \frac{Px}{48EI}(3l^2 - 4x^2)$

AISC Manual

4. Superposition of Equations

Equations of shear or moment may be combined (superimposed) for any number of cases.

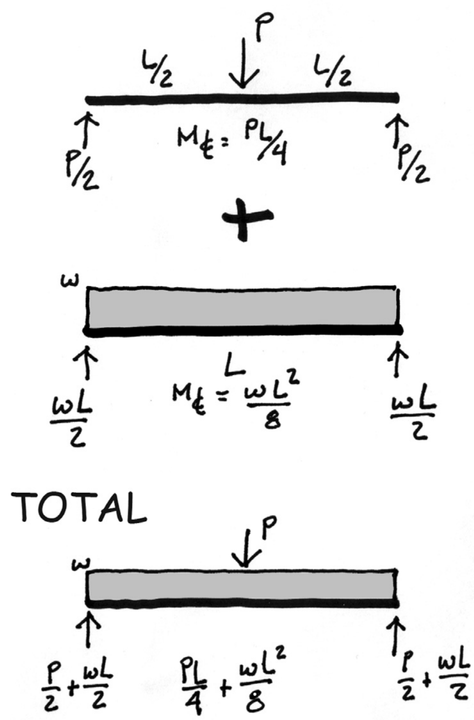
BUT

The appropriate location along the beam for which the equation is valid must be maintained

Thus

At the reaction, V = P/2 + wL/2

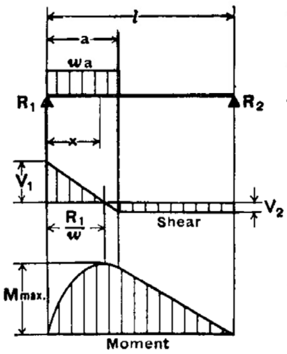
And at the C.L. M = PL/4 + $wL^2/8$



Non-symmetric

For more complex loads, care must be taken to combine equations at the same location or point on the beam (x).

5. SIMPLE BEAM—UNIFORM LOAD PARTIALLY DISTRIBUTED AT ONE END



$R_1 = V_1$ max. . . . . $= \frac{wa}{2l}(2l-a)$

$R_2 = V_2$ . . . . $= \frac{wa^2}{2l}$

V_x (when $x < a$) . . . . . . $= R_1 - wx$

M max. (at $x = \frac{R_1}{w}$) . . . . . . $= \frac{R_1^2}{2w}$

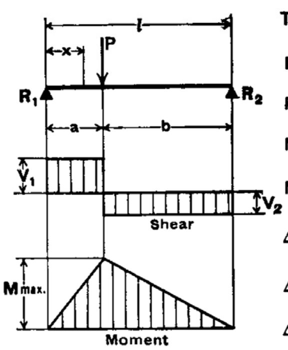
M_x (when $x < a$) . . . . . . $= R_1 x - \frac{wx^2}{2}$

M_x (when $x > a$) . . . . . . $= R_2 (l - x)$

Δx (when $x < a$) . . . . . . $= \frac{wx}{24EIl}\left(a^2(2l-a)^2 - 2ax^2(2l-a) + lx^3\right)$

Δx (when $x > a$) . . . . . . $= \frac{wa^2(l-x)}{24EIl}(4xl - 2x^2 - a^2)$

8. SIMPLE BEAM—CONCENTRATED LOAD AT ANY POINT



Total Equiv. Uniform Load . . . . . . $= \frac{8Pab}{l^2}$

$R_1 = V_1$ (max. when $a < b$) . . . . $= \frac{Pb}{l}$

$R_2 = V_2$ (max. when $a > b$) . . . . $= \frac{Pa}{l}$

M max. (at point of load) . . . . $= \frac{Pab}{l}$

M_x (when $x < a$) . . . . $= \frac{Pbx}{l}$

Δmax. (at $x = \sqrt{\frac{a(a+2b)}{3}}$ when $a > b$) $= \frac{Pab(a+2b)\sqrt{3a(a+2b)}}{27EIl}$

Δa (at point of load) . . . . $= \frac{Pa^2b^2}{3EIl}$

Δx (when $x < a$) . . . . $= \frac{Pbx}{6EIl}(l^2 - b^2 - x^2)$

4. Superposition of Equations - example

find x at M_{max} for combined asymmetric cases

END REACTIONS (NOTE: VARIABLES DIFFER IN DIFFERENT EQUATIONS)

$R_L = \frac{wa}{2l}(2l - a) + \frac{Pb}{l}$

$R_L = \frac{2(10)}{2(20)}(2(20) - 10) + \frac{8(5)}{20}$

$R_L = 15 + 2 = 17^K$

MOMENT EQUATIONS AT x

$M_x = R_1 x - \frac{wx^2}{2} + \frac{Pbx}{l}$

TO FIND x: DIFFERENTIATE AND SOLVE AT 0 (M max)

$R_1 - wx + \frac{Pb}{l}$

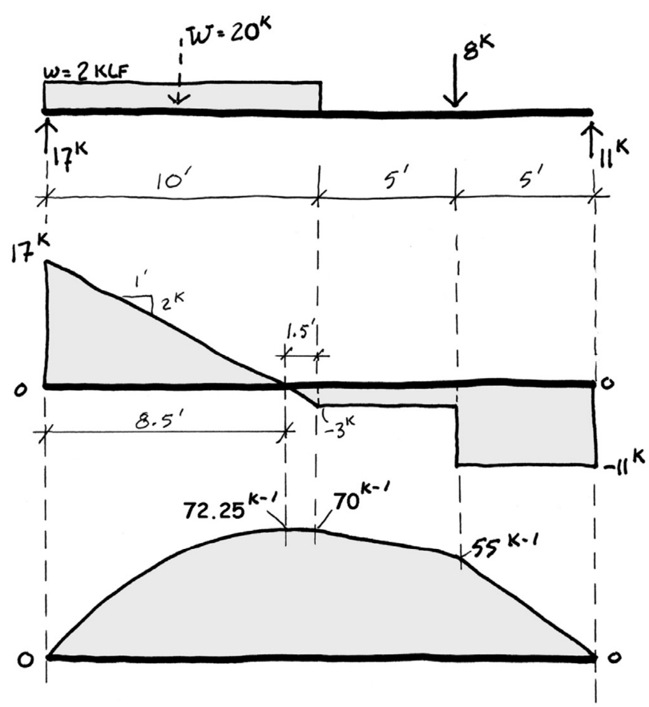
$15 - 2x + 2 = 0$

$x = 8.5'$

AND

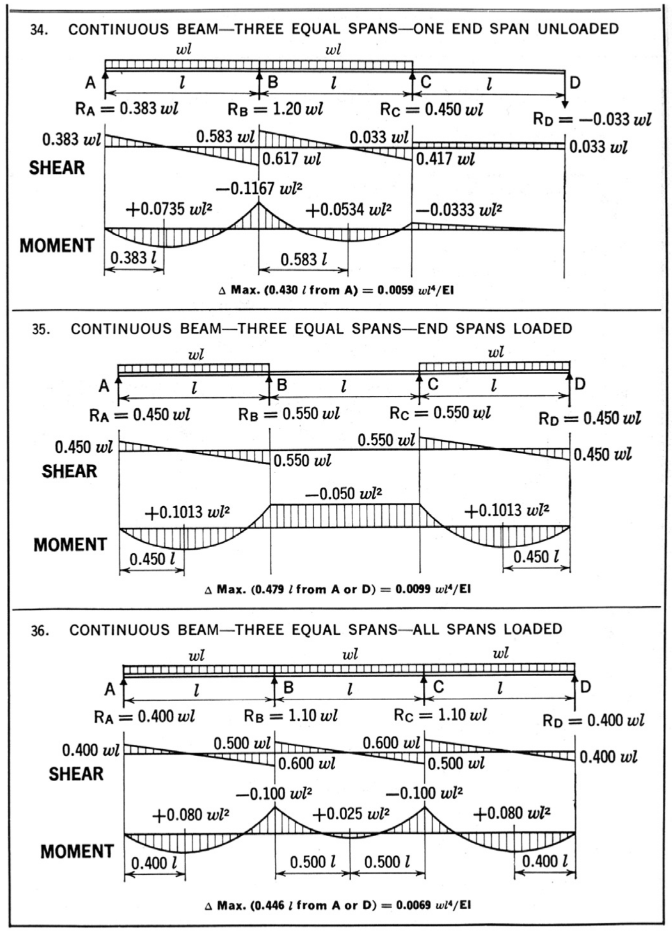
$M_{max} = 15(8.5) - \frac{2(8.5)^2}{2} + \frac{8(5)(8.5)}{20}$

$M_{max} = 55.25 + 17 = 72.25^{K\text{-}'}$



Simple vs. Continuous Beams

- Simple Beam
  - End moments = 0
  - when symmetric, M_{max} at C.L. e.g. $wL^2/8 = 0.125wL^2$
- Continuous Beam
  - Exterior end moments = 0
  - Interior support moments are usually negative
  - Mid-span moments are usually positive
  - End + Mid = $0.125wL^2$



Note: moments shown reversed

Moment Diagram vs. Catenary Curve

For a gravity loaded simple span beams, the shape of the of the moment diagram is the inverse of the catenary curve.